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RR 077

SEPTEMBER 1989

RESEARCH REPORTS

DEPARTMENT OF MATHEMATICS

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Key results in this paper have subsequently been published with new exposition in "Naturman, Colin Ashley. Interior algebras and topology. Diss. University of Cape Town, 1991, Chap 1 Basic Theory of Interior Algebras & Chap 2 Topological Duality"
This paper has also been republished as the journal article "Naturman, Colin, and Henry Rose. Ideal algebras of interior algebras. Ordered Sets and Lattices 11 (1995): 39-44."

R E S E A R C H R E P O R T S

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*Printed in the Department of Mathematics, University of Cape Town, Private Bag,
Rondebosch 7700, Republic of South Africa.*

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Abstract

An interior algebra is a Boolean algebra enriched with an interior operator. Given an interior algebra there is a natural way of forming interior algebras from its principal ideals. Basic results concerning these ideal algebras, Stone spaces of ideal algebras and preservation properties of ideal algebras are investigated.

1980 Mathematics Subject Classification: 06F99, 06E99

¹ The work of the second author was supported by a grant from the University of Cape Town Research Committee, and the Topology Research Group from the University of Cape Town and the South African Council for Scientific and Industrial Research.

INTRODUCTION

Given an interior algebra B we can form interior algebras from the principal ideals of B in a natural way. These ideal algebras are a generalization of the quotients of B by open elements. (See [2] and Proposition 1.4.) Ideal algebras are also an algebraic generalization of topological subspaces (see Corollary 1.3).

Ideal algebras were introduced in [1] under the name "relativised subalgebras" using a different approach to the one we use. Proposition 1.3 shows that our definition and the one in [1] are equivalent. Very few results concerning ideal algebras were discussed in [1] and no further results, except those in [2], have been published since [1]. In Section 1 of this paper we investigate some basic results concerning ideal algebras. In Section 2 Stone spaces of ideal algebras are investigated and in Section 3 we examine the preservation properties of ideal algebras. $\square$

NOTATION AND TERMINOLOGY

We use the same terminology and notational conventions as in [2] and we assume that the reader is familiar with these. In addition if B is an interior algebra and a is any element of B , not necessarily open, then B/a denotes the principal ideal of B generated by a . $\square$

1. BASIC RESULTS CONCERNING IDEAL ALGEBRAS

Lemma 1.1

Let B be an interior algebra and let $a \in B$. Define operations ^I_a and $/_a$ on B/a by:

$$b^I_a = a(a' + b)^I, \quad b/_a = ab'$$

for all $b \in B/a$.

Then $B/a = \langle B/a, \cdot, +, /_a, ^I_a, 0, a \rangle$ is an interior algebra.

Proof:

Firstly note that B/a is closed under $\cdot, +$ and $0, a$ are the bottom and top of B/a respectively. Also $/_a$ is clearly a complementation on B/a . It remains to show that ^I_a is an interior operator on B/a .

1. For all $b \in B/a$

$$b^I_a = a(a' + b)^I \leq a(a' + b) = ab \leq b.$$

2. For all $b \in B/a$

$$(a' + b)^I \leq a' + (a' + b)^I = a' + a(a' + b)^I = a' + b^I_a.$$

Hence $(a' + b)^I \leq (a' + b^I_a)^I$ and so $a(a' + b)^I \leq a(a' + b^I_a)^I$ i.e.

$$b^I_a \leq b^{I_a I_a}. \text{ But } b^{I_a I_a} \leq b^I_a \text{ and so } b^I_a = b^{I_a I_a}.$$

3. For all $b, c \in B/a$

$$\begin{aligned} b^I_a c^I_a &= a(a' + b)^I a(a' + c)^I = a(a' + b)^I (a' + c)^I = a[(a' + b)(a' + c)]^I \\ &= a(a' + bc)^I = (bc)^I_a. \end{aligned}$$

4. $a^I_a = a(a' + a)^I = a.1 = a$.

Thus ^I_a is an interior operator on B/a as required. $\square$

Definition 1.2

Let B be an interior algebra and $a \in B$. Then the interior algebra B/a is called the **ideal algebra** of B generated by a . $\square$

From now on B will denote an interior algebra and a will denote a fixed element of B . ^{ca} will denote the closure operation of B/a .

Proposition 1.3

For all $b \in B/a$, $b^{ca} = abc$.

Proof:

Let $b \in B/a$. Then $b^{ca} = b/a^{Ia}/a = (ab)^{Ia}/a = [a(a' + ab')^I]^I/a = a[a(a' + ab')^I]^I = a[a' + (a' + ab')^I]^I = a(a' + ab')^I = a[a(a' + b)]^c = a(ab)^c = abc$. $\square$

Proposition 1.4

If a is open then for all $b \in B/a$, $b^{Ia} = b^I$.

Proof:

Let $b \in B/a$. Then $b^{Ia} = a(a' + b)^I = a^I(a' + b)^I = [a(a' + b)]^I = (ab)^I = b^I$. $\square$

Proposition 1.5

- (i) $L(B/a) = \{ab : b \in L(B)\}$
- (ii) $L'(B/a) = \{ab : b \in L'(B)\}$.

Proof:

- (i) If $c \in L(B/a)$ then $c = c^{Ia}$ i.e. $c = a(a' + c)^I$, and $(a' + c)^I \in L(B)$.
Conversely if $b \in L(B)$ then $(ab)^{Ia} = a(a' + ab)^I = a(a' + b)^I \geq ab^I = ab$.
But $(ab)^{Ia} \leq ab$ and so $(ab)^{Ia} = ab$ whence $ab \in L(B/a)$.
- (ii) If $c \in L'(B/a)$ then $c = c^{ca} = ac^c$ and $c^c \in L'(B)$. Conversely if $b \in L'(B)$ then $(ab)^{ca} = a(ab)^c \leq abc^c = ab$. But $ab \leq (ab)^{ca}$ and so $(ab)^{ca} = ab$ whence $ab \in L'(B/a)$. $\square$

Corollary 1.6

- (i) If a is open $L(B/a) = L(B) \cap B/a$.
- (ii) If a is closed $L'(B/a) = L'(B) \cap B/a$. $\square$

Corollary 1.7

Let X be a topological space and let Y be a subspace of X . Then

$$A(X)/Y = A(Y).$$

Proof:

Clearly $Ba A(X)/Y = Ba A(Y)$. By Proposition 1.5 (i) $L(A(X)/Y) = LA(Y)$ and so the result follows. $\square$

We mention an important result which was proved in [2].

Proposition 1.8

If a is open the map $b \rightarrow ab$ is an epimorphism from B to B/a . Consequently $B/a \cong B/\theta_{[a]}$. $\square$

Thus the ideal algebras of B generated by its open elements are, up to isomorphism, just the principal quotients of B . (See Section 5 of [2].)

2. STONE SPACES OF IDEAL ALGEBRAS

Recall that $\alpha(a) = \{F \in T(B) : a \in F\}$. We thus have a subspace $\alpha(a)$ of $T(B)$.

Proposition 2.1

$\alpha[L(B/a)]$ is a base for $\alpha(a)$.

Proof:

$\alpha[L(B)]$ is a base for $T(B)$ and so a base for $\alpha(a)$ is given by

$\{\alpha(a) \cap \alpha(b) : b \in L(B)\} = \{\alpha(ab) : b \in L(B)\} = \alpha[L(B/a)]$, by Proposition 1.5 (i). $\square$

Lemma 2.2

If $F \in \alpha(a)$ then $F \cap B/a \in T(B/a)$.

Proof:

Since $F \in \alpha(a)$, $a \in F \cap B/a$. Consider $b, c \in B/a$. If $b \leq c$ and $b \in F \cap B/a$ then $b \in F$ and so $c \in F$. Hence $c \in F \cap B/a$. If $b, c \in F \cap B/a$ then $b, c \in F$ whence $bc \in F$. But $bc \in B/a$ and so $bc \in F \cap B/a$. Thus $F \cap B/a$ is a filter in B/a .

Consider $b \in B/a$. If $b \notin F \cap B/a$ then $b \notin F$ whence $b' \in F$. Now $a \in F$ and so $b/a = ab' \in F$. Hence $b/a \in F \cap B/a$. $\square$

Lemma 2.3

If $G \in T(B/a)$ there is an $F \in T(B)$ such that $F \cap B/a = G$.

Proof:

Since $G \in T(B/a)$, G has the finite intersection property and so there is an $F \in T(B)$ with $G \subseteq F$. Now $a \in G \subseteq F$ and so by Lemma 2.2 $F \cap B/a \in T(B/a)$. Now $G \subseteq F \cap B/a$ and so, since a and $F \cap B/a$ are ultrafilters $a = F \cap B/a$. $\square$

By Lemma 2.2 we can define a map $p : \alpha(a) \rightarrow T(B/a)$ by $p(F) = F \cap B/a$ for all $F \in \alpha(a)$. By Lemma 2.3 p is surjective.

Let α^a denote the α map for B/a .

Lemma 2.4

For $b \in B/a$, $p[\alpha(b)] = \alpha^a(b)$ and $p^{-1}[\alpha^a(b)] = \alpha(b)$.

Proof:

Let $F \in \alpha(b)$. Then $b \in F$ and so, since $b \leq a$, $b \in p(F)$ i.e. $p(F) \in \alpha^a(b)$.

Conversely if $G \in \alpha^a(b)$ there is an $F \in T(B)$ with $p(F) = G$. Then $b \in p(F) \subseteq F$ and so $F \in \alpha(b)$ whence $G \in p[\alpha(b)]$. Thus $p[\alpha(b)] = \alpha^a(b)$.

Let $F \in p^{-1}[\alpha^a(b)]$. Then $p(F) \in \alpha^a(b)$ whence $b \in p(F) \subseteq F$. Thus $F \in \alpha(b)$.

Conversely if $F \in \alpha(b)$, $b \in F$ and so $b \in p(F)$ since $b \in B/a$. Thus $p(F) \in \alpha^a(b)$ i.e. $F \in p^{-1}[\alpha^a(b)]$. Thus $p^{-1}[\alpha^a(b)] = \alpha(b)$. $\square$

Theorem 2.5

- (i) $p : \alpha(a) \rightarrow T(B/a)$ is a continuous open surjection, in particular $T(B/a)$ is a quotient of $\alpha(a)$.
- (ii) If a is of finite height in B then $p : \alpha(a) \rightarrow T(B/a)$ is a homeomorphism.

Proof:

- (i) This follows from Proposition 2.1 and Lemma 2.4.
- (ii) Let a be of finite height. Let $F, G \in \alpha(a)$ with $p(F) = p(G)$. Then there is an atom b of B/a such that $p(F), p(G)$ are both the principal filter in B/a generated by b . Then b is an atom in B and so both F and G must be the principal filter in B generated by b whence $F = G$. Thus p is injective and so by (i) p is a homeomorphism. $\square$

Corollary 2.6

- (i) B/a is embeddable in $A(\alpha(a))$.
- (ii) If a is of finite height in B then $B/a \cong A(\alpha(a))$.

Proof:

- (i) By Theorem 2.5 (i) $Ap : AT(B/a) \rightarrow A(\alpha(a))$ is an embedding. Since $\alpha^a : B/a \rightarrow AT(B/a)$ is an embedding the result follows.
- (ii) By Theorem 2.5 (ii) $Ap : AT(B/a) \rightarrow A(\alpha(a))$ is an isomorphism. Since a is of finite height B/a is finite and so $\alpha^a : B/a \rightarrow AT(B/a)$ is an isomorphism. $\square$

3. PRESERVATION PROPERTIES OF IDEAL ALGEBRAS

Proposition 3.1

Let $\{B_i : i \in I\}$ be a family of interior algebras and let $a_i \in B_i$ for all $i \in I$. Then

$$\Pi_I (B_i/a_i) = (\Pi_I B_i)/a$$

where $a = (a_i)_I$.

Proof:

For $b = (b_i)_I \in \Pi_I B_i$ we have $b \leq a$ iff $b_i \leq a_i$ for all $i \in I$ and so

$\Pi_I (B_i/a_i) = (\Pi_I B_i)/a$. Using the componentwise definition of operations on a product it is not difficult to see that the operations on $\Pi_I (B_i/a_i)$ and $(\Pi_I B_i)/a$ coincide. $\square$

Proposition 3.2

Let $f : B \rightarrow C$ be an homomorphism/embedding/epimorphism, then $f|_{B/a} : B/a \rightarrow C/f(a)$ is a homomorphism/embedding/epimorphism respectively. $\square$

Corollary 3.3

Let $\mathcal{A} = \{f_{ij} : B_i \rightarrow B_j : i \leq j, i, j \in I\}$ be a directed system of homomorphisms and let $\{f_i : B \rightarrow B_i : i \in I\}$ be the inverse limit of $\mathcal{A}$. Let $a \in B$ and for all $i \in I$ put $a_i = f_i(a)$.

Then $\mathcal{A}/a = \{f_{ij}|_{B_i/a_i} : B_i/a_i \rightarrow B_j/a_j : i \leq j, i, j \in I\}$ is a directed system with inverse limit $\{f_i|_{B/a} : B/a \rightarrow B_i/a_i : i \in I\}$. $\square$

Proposition 3.4

Let $b \leq a$. Then $B/b = (B/a)/b$.

Proof:

Obviously $B/b = (B/a)/b$. The complementation in $(B/a)/b$ is given by $bd/a = bad' = bd' = d/b$ for all $d \leq b$. The interior operator in $(B/a)/b$ is given by $b(b' + d)^I = ba[a' + (b' + d)]^I = b(b' + d)^I = d/b$ for all $d \leq b$.

Thus $B/b = (B/a)/b$. □

Another preservation result which follows easily from Proposition 1.5 (i) is :

Proposition 3.5

If B is simple then B/a is simple. □

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